A Robust Estimation of the Portuguese Real Business Cycles

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Abstract

The application of the Hodrick-Prescott (HP) and other linear filters to remove trend and extract business cycles in macroeconomic time series is a common practice despite its limitations, namely, in signaling recessions. Median filters and other nonlinear techniques can perform better by accommodating sharp but fundamental changes in the growth trend and passing only the relevant information to the cycle component. An application to the Portuguese relevant macroeconomic series confirmed the robustness of nonlinear filters in signaling the recessions and recoveries. In particular, the Mosheiov-Raveh (MR) filter estimates piecewise trend growth paths that naturally date the specific periods of the Portuguese economy since 1977.

Keywords: Time series models; Business cycles; Linear and nonlinear filtering.

JEL codes: C22, E32

1 Introduction

Many economic time series like GDP increase steadily over time. A simple way to describe such upward trends is to consider a linear model of the data y_t on a deterministic time trend plus a noise component [7, 9, 10]:

$$y_t = \alpha + \delta t + \varepsilon_t. \tag{1}$$

Processes with such representation are typically described as *trend stationary*, in the sense that if one subtracts the trend $\alpha + \delta t$ from (1), the result is

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a stationary process, that is, whose mean and autocovariances do not depend on the date t. However, this regression technique may not be appropriate to detrend series whose growth component is time varying, depends on other endogenous or exogenous conditions, or is stochastic. In fact, macroeconomic time series are frequently represented by the random walk plus drift model, a special case of unit root process:

$$y_t = y_{t-1} + \delta + \varepsilon_t. \tag{2}$$

Given the initial condition y_0, the general solution for this difference equation is [7]:

$$y_t = y_0 + \delta t + \sum_{i=1}^{t} \varepsilon_t. \tag{3}$$

Thus, detrending a process such (2) is not sufficient to eliminate the stochastic trend $\sum \varepsilon_t$ where each shock ε_i with $i \leq t$ has a permanent effect on the mean of y_t. Nevertheless, its representation in first-differences:

$$\Delta y_t \equiv (1 - L)y_t = y_t - y_{t-1} = \delta + \varepsilon_t \tag{4}$$

where L is the lag operator is stationary. Thus, the random walk with drift is an example of a *difference stationary* model that can be transformed into a stationary process by differencing [7].

Detrending and differencing are popular practices in macroeconometrics because the autoregressive moving average (ARMA) framework is applicable only to stationary time series. However, both practices have several setbacks. On one hand, the linear time trend model (1) assumes that trend is unchanging over time with a secular growth rate δ that could not accommodate technological, demographic and other fundamental shocks that might occur overtime. On the other hand, the first-difference filter (4) removes the zero (long-run) frequency and accentuates the high-frequencies in the data y_t, producing a very noisy stationary component ε_t [5, 10].

In practice, detrending and differencing may not be adequate to describe the *business cycle*, that is, the more or less regular pattern of expansion and contraction in economic activity around the path of trend growth that is observed in most macroeconomic variables. Given a seasonally adjusted time series y_t, the business cyclical component c_t can be defined as

$$c_t \equiv y_t - x_t - \epsilon_t, t = 1, \ldots, N \tag{5}$$

where x_t and ϵ_t are the trend (low-frequency) and irregular (high-frequency) components of y_t, respectively. Thus, the cycle c_t should capture the medium-frequencies with periods lasting no fewer than six and no more than thirty-two quarters as defined by the National Bureau of Economic Research (NBER) researchers [22].

In most business cycle applications, the estimation of the low-frequency component x_t is the key task, with the cycle c_t being approximated by the residual $\varepsilon_t^* = y_t - x_t^*$. The trend is generally seen as a slow movement over time that could be extracted with appropriate filtering techniques. Ideally, they should capture not only the deterministic but also the stochastic dimension of the growth as described. In several cases, filtering consists in applying a smooth operator M to a moving-centered window of the data with length $l = 2k + 1$ in order to estimate the trend component

$$x_t = \mathrm{M}\left[y_{t-k}, \ldots, y_t, \ldots, y_{t+k}\right], t = 1, \ldots, N. \tag{6}$$

Linear and nonlinear techniques can be mobilized to perform this task. Moving Average (MA), that is, using the arithmetic mean as smooth operator in equation (6) is a simple way to do it:

$$\tilde{x}_t = \frac{1}{2k+1} \sum_{i=-k}^{k} y_{t+i}, t = 1, \ldots, N. \tag{7}$$

This is an example of a Linear Time-Invariant (LTI) low-pass filter which does not affect low frequencies, rejects high-frequencies and avoid phase shifts through its symmetry [21]. Setting $k = 11$, the MA filter can be used to block cycles with length shorter than $l = 23$ quarters [23]. And it can be applied once again to the residual $\tilde{\varepsilon}_t = y_t - \tilde{x}_t$ to separate the pure cycle component c_t from the high-frequencies ϵ_t by fixing $k = 2$ (window length of 5 quarters) as in the decomposition (5), even when the trend x_t was estimated with other techniques including the linear time trend regression (1).

A special case of moving average and linear filtering is the widely used Hodrick-Prescott (HP) filter [13]. In this method, the trend sequence $\{x_t\}$ is chosen to minimize either the sum of the squares of deviations from data series y_t or the sum of the squares of the trend's second difference, a possible measure of its smoothness. The benefit of the HP filter is that it can extract the same stochastic trend from a set of variables [7].

In a recent critique, Hamilton [11] stressed that the HP filter produces extremely predictable cyclical components whose rich dynamics are purely artifacts created by the filter rather than reflecting any true dynamics of the data-generating process itself, following an argument similar to Harvey and Jaeger [12] that found that the HP filter may create spurious cycles and cross-correlations between different variables. This is a common feature of linear filters including the symmetric Baxter-King (BK) [1] and asymmetric Christiano-Fitzgerald (CF) [4] band-pass filters which tend to eliminate large fluctuations either in expansions or contractions [23]. In fact, linear filters produce smooth trends that might not capture fundamental sharp changes in the growth component of the time series under study. This problem is particularly acute with the HP method because it produces a trend with a very regular, predictable, pattern and passes everything else to the cyclical component, including high-frequencies.

As suggested by Wen and Zeng [23], nonlinear filters could perform better than linear filters in capturing occasional, discrete shifts in the growth dynamics of economic series. In particular, the Median (MED) filter provides a simple noise attenuation with robustness against impulsive-type noise and it has proved to be effective in signaling recessions. However, it produces a very noisy trend that might fail the test of either smoothness or persistence. The Mosheiov-Raveh (MR) [18] alternative metric for the HP minimization problem could deal with this problem, producing a suggestive piecewise trend.

We start by describing these different filtering techniques (MED, HP and MR). Then, we compare the different insights in describing the business cycles of the Portuguese economy since 1977. One conclusion of our analysis is that the trend and the cyclical component are both informative about economic conditions. As framework, we dated the Portuguese contractions (recessions) and expansions (recoveries) by applying the Bry and Boschan algorithm [2, 14] to GDP, private consumption, investment and employment (see appendix).

2 Alternative filtering techniques

2.1 The Median (MED) filter

Linear filters implicitly assign all sharp changes in time series to shifts in non-fundamentals, assuming away the possibility that the growth trend can also experience sudden shifts or jumps. In fact, linear filters can only suppress undesired parts of the signal and retain the desired information about business cycles if and only if the noise and the signal occupy different frequency bands. In reality, however, signals reflecting sudden but fundamental changes could share the same frequency band with noise.

To deal with these limitation of linear filters, Wen and Zeng [23] introduced a class of nonlinear filters, called the *median filters*, that has been proven useful and powerful for removing time trend and noise in several contexts. To compute the output of the Median (MED) filter, an odd number of sample values are sorted, and the middle or median value is used as the trend component x_t. In practice, the MED filter adopted the median X as smooth operator M in equation (6):

$$\hat{x}_t = \mathrm{X}\left[y_{t-k}, \ldots, y_t, \ldots, y_{t+k}\right] \tag{8}$$

for $t = 1, \ldots, N$ and $k = 11$ with quarterly data. To be able to filter also the outermost observations, where the filter window partially fall outside the input signal, those authors [23] replicated the y_1 and y_N values as many times as needed, the so-called 'first and last values carry-on appending strategy'.[1]

The MED output has good deterministic and statistical properties [23]. Namely, it is optimal in the mean absolute error sense because the median of $y_1, \ldots, y_N$ can be defined as the value β minimizing the following expression when $\gamma = 1$

$$\sum_{t=1}^{N} |y_t - \beta|^{\gamma} \tag{9}$$

In fact, the sample median is the maximum likelihood estimate for the location parameter of the Laplace probability distribution [23].

Wen and Zeng [23] compared the MED filter with length $l = 2k + 1 = 23$ quarters with the HP, BK and MA filters and concluded that the MED cycle $\hat{c}_t \equiv y_t - \hat{x}_t$ for GDP coincides almost exactly with the NBER dated recessions for the United States of America, while the linear filters are too noisy outside contraction periods. This result indicates that linear filters are not efficient in retaining the growth component of GDP in terms of effectively capturing sharp changes in growth trend. Those authors also stressed that the HP and

[1] We follow that strategy in the application of the moving average filter (7) too.

MA filters generate very similar trend paths which could be explained by the moving average representation of the HP filter, as described below.

2.2 The Hodrick-Prescott (HP) filter

Let denote $y = (y_1, y_2, \ldots, y_N)$ as the $(N \times 1)$ vector of the seasonally adjusted time series y_t. The Hodrick-Prescott (HP) trend $x^* = (x_1, x_2, \ldots, x_N)$ is chosen to minimize either the sum of the square residuals $\varepsilon_t = y_t - x_t$ or the smoothness of the trend component

$$\min \left[\sum_{t=1}^{N} \varepsilon_t^2 + \lambda \sum_{t=3}^{N} (g_t - g_{t-1})^2 \right] \tag{10}$$

where $\lambda > 0$ is a penalty for the square of the difference of the trend growth $g_t \equiv x_t - x_{t-1}$, which is the second difference (acceleration) of the trend component x_t. Thus, the larger the value of λ, the smoother will be the HP trend. In particular, as λ approaches infinity, the limit of solutions to program (10) is the least squares fit of the linear time trend model (1). If the residual ε_t and the difference $g_t - g_{t-1}$ are uncorrelated white noise processes with means zero and variances σ_ε^2 and σ_g^2, then the conditional expectation of x_t on data y_t would be the solution to program (10) when $\lambda = \sigma_\varepsilon^2/\sigma_g^2$ [6, 11, 13]. In order to extract business cycles frequencies from quarterly data, Hodrick and Prescott [13] propose setting this noise-to-signal ratio equal to 1600.

It is convenient to express the problem (10) in matrix form

$$\min \left[(y - x)^T (y - x) + \lambda x^T D^T D x \right] \tag{11}$$

where $D \in \mathbb{R}^{(N-2) \times N}$ is an upper triangular Toeplitz matrix with first row $[1 \ -2 \ 1 \ 0 \cdots 0]$. As noted by Kim *et al.* [15], this objective function is strictly convex in x, thus has a unique minimizer

$$x^* = \left(I + \lambda D^T D \right)^{-1} y. \tag{12}$$

From the optimality condition $y - x^* = \lambda D^T D x^*$, we could obtain the optimal fitting error, that is, the cyclical component of the HP filter

$$c^* \equiv y - x^* = \lambda D^T D \left(I + \lambda D^T D \right)^{-1} y. \tag{13}$$

From the first order condition (12), the HP trend estimate is simply a moving average of the original unfiltered data y whose weights change as we move from the mid-sample to the end (or the begin) of the sample. As stressed by St-Amant and van Norden [22], the HP filter produces a smooth two-sided average at mid-sample where no observation receives more than 6 percent of the weight. Nevertheless, at the end of sample, the last observation alone counts for 20 percent of the weight used to compute x_N^*, and even at end–3 periods sample it counts 13.5 percent for x_{N-3}^*. Symmetrically, this problem is also observed at the begin of the sample.

The HP filter is an example of an heuristic two-part decomposition of time series that could be represented by a high-pass transfer function. As found by King and Rebelo [16], the transfer function of the cycle component (13) is given by

$$H(\omega;\lambda) = \frac{4\lambda(1-\cos(\omega))^2}{1+4\lambda(1-\cos(\omega))^2} \tag{14}$$

where the frequency ω measures the number of cycles completed during 2π periods which is unity for the cosine function. The HP transfer function is also the gain of the filter because it assumes only *real* (non imaginary) values. This result is a direct consequence of the symmetry of the HP filter and from Euler equations (see also [10]). Thus, the HP filter is similar to a high-pass filter where choosing different values for λ is comparable to fix different values for the cut-off point of the filter [19].

Solving in λ the equation (14) for a gain of 0.5 as suggested by Mohr [17], we found that the critical frequency of $\pi/20 = 2\pi/40 = 0.157$, which corresponds to a period of 40 quarters (10 years), requires a value of $\lambda \approx 1600$ as suggested by Hodrick and Prescott [13] to filter quarterly data. Moreover, for a gain of 0.7, the critical frequency associated with $\lambda = 1600$ is approximately $\pi/16$, the cut-off frequency of a filter that passes oscillations lasting 32 quarters (8 years) or less. Thus, the HP cyclical component of quarterly data computed with $\lambda = 1600$ shall have residual seasonality and other high frequencies with period less than 6 quarters mixed with real business cycle frequencies. Applying the MA filter (7) with $k = 2$ ($l = 5$) to the HP cycle (13) can mitigate this problem.

2.3 The Mosheiov-Raveh (MR) filter

As suggested above, the median filter output minimizes the sum of the absolute values of the fitting residual $\varepsilon_t = y_t - x_t$. Thus, the MED trend $\hat{x_t}$ is well-fitted to the original time series y_t and it is always one of the input observations from (8). The fidelity or closeness to data is an important property of the trend but the smoothness should also be considered. So, we could imagine a generalized median filter which results from the following problem

$$\min\left[\sum_{t=1}^{N}|y_t - x_t| + \theta\sum_{t=1}^{N-2}|(x_{t+2}-x_{t+1})-(x_{t+1}-x_t)|\right] \tag{15}$$

where $\theta = (1-\alpha)/\alpha$ with $0 \le \alpha \le 1$ is the weight assigned to the smoothness criterion. If $\theta = 0$ ($\alpha = 1$) then the solution of the problem (15) is the MED trend $\hat{x_t}$.

This objective function, originally proposed by Mosheiov and Raveh (MR) [18], is a variation of the HP objetive function (10) which substitutes the sum of squares (that is, the squared l_2 Euclidean norm) for the sum of absolute values (the l_1 norm) to measure either the fidelity or the smoothness of the trend x_t. Mosheiov and Raveh suggested $\alpha = 0.1$, that is, $\theta = 9$ but a more natural choice for quarterly data might be $\theta = \sqrt{1600} = 40 = 5/(1/8)$, recalling that Hodrick and Prescott [13] fixed $\lambda = 1600$ by assuming that a 5 percent cyclical component is considered moderately large, as a one-eighth of 1 percent change in the growth rate in a quarter.

To eliminate the absolute values in objective function (15), Mosheiov and Raveh [18] proposed a linear programming approach with an appropriate change of variable

$$\begin{array}{lll}
\min & \left[\sum_{t=1}^{N}(u_t+v_t)+\theta\sum_{t=1}^{N-2}(a_t+b_t)\right] & \\
& & \\
\text{s.t.} & u_t-v_t=y_t-x_t & t=1,\ldots,N \\
& a_t-b_t=x_{t+2}-2x_{t+1}+x_t & t=1,\ldots,N-2 \\
& x_t\leq x_{t+1} & t=1,\ldots,N-1 \\
& u_t,v_t,x_t\geq 0 & t=1,\ldots,N \\
& a_t,b_t\geq 0 & t=1,\ldots,N-2.
\end{array} \tag{16}$$

The variables u_t and v_t have a very intuitive interpretation. They represent either a positive or a negative cyclical position, and therefore at least one of them has to be equal to zero. Similarly, the variables a_t and b_t represent a sort of regime change where the economy either accelerates to a higher trend growth level or decelerate to a lower one. Likewise, they cannot be both positive. It is this interesting characteristic of the optimal solution to the MR linear program that allows this model to reveal the natural growth periods present in the data.

The problem (15) assumes monotonicity of the trend component x_t. However, the Portuguese data shows a significant number of time periods with negative trend growth throughout data series. Therefore, one has to find an adequate transformation of the original data that keeps the robust proprieties of the MR filter while allowing the data to reveal both negative or positive trend growth.

So, we propose the following transformation of the original data y_t:

$$y'_t=y_t+\vartheta(t-1), t=1,\ldots,N, \tag{17}$$

where $\vartheta>0$ is the symmetric of the lower bound of the growth trend g_t when it is negative.[2] Thus, the MR linear program becomes

$$\begin{array}{lll}
\min & \left[\sum_{t=1}^{N}(u'_t+v'_t)+\theta\sum_{t=1}^{N-2}(a'_t+b'_t)\right] & \\
& & \\
\text{s.t.} & u'_t-v'_t=y'_t-x'_t & t=1,\ldots,N \\
& a'_t-b'_t=x'_{t+2}-2x'_{t+1}+x'_t & t=1,\ldots,N-2 \\
& x'_t\leq x'_{t+1} & t=1,\ldots,N-1 \\
& u'_t,v'_t,x'_t\geq 0 & t=1,\ldots,N \\
& a'_t,b'_t\geq 0 & t=1,\ldots,N-2.
\end{array} \tag{18}$$

One now can reconstruct the MR trend for the original data using the estimated values of x'_t in the following way:

$$\hat{\hat{x}}_t=x'_t-\vartheta(t-1), t=1,\ldots,N. \tag{19}$$

The transformation (17) of the data series is invariant in the following sense: given a data series y_t and a smooth parameter θ, if the solution of the program (16) provides $x_{t+1}-x_t$ that are all strictly positive, than the solution to the program (18) yields the same estimate of the trend after applying equation (19). That is, $\hat{\hat{x}}_t=x_t$ for all t. Therefore, the proposed transformation is invariant for all series with a positive trend. When the non-negative condition on program

[2]We used as this bound the lowest observed trend growth given by the HP filter (12) for the relevant series. For Portuguese GDP, the assumed ϑ parameter is 0.33.

(16) is binding, the solution $\hat{\hat{x}}_t$ is obviously different. But that is exactly the purpose of the transformation, allowing for a non-monotone trend series which can be represented by periods with positive or negative trend growth.

With this additional procedure, the MR filter produces good estimates, including a possible negative trend growth. It also preserves the good properties of the MED filter with a trend estimate that is smooth in the sense of being piecewise linear [15]. In addition, the MR filter does not use fixed weights or coefficients as HP or MA filters.

2.4 Filter proprieties

The reason for the popularity of the HP filter among practitioners rests on a few properties. First, it smooths the data. That is common to all filters, except the linear filter that smooths the data too much. Second, it allows for an estimate of the trend growth. Once again, this is common feature to all statistical filters. Third, it provides an estimate of the cyclical position of an economy in every quarter or year. This is also common to all filters, even though the linear filter accepts too large cyclical components, and the HP and the other filters may underestimate the cyclical component. This is one important advantage of nonlinear filters, providing a potentially better measure of the cyclical position and as such a measure of the slack or overheating of the economy. Fourth, the key distinctive feature of the HP filter is that it allows trend growth to be different in every time period. In this perspective, the HP filter is the simplest statistical method that assumes non-constant GDP growth. Moreover, it allows this growth to follow the data, even though in much smoother form than actual observed growth.

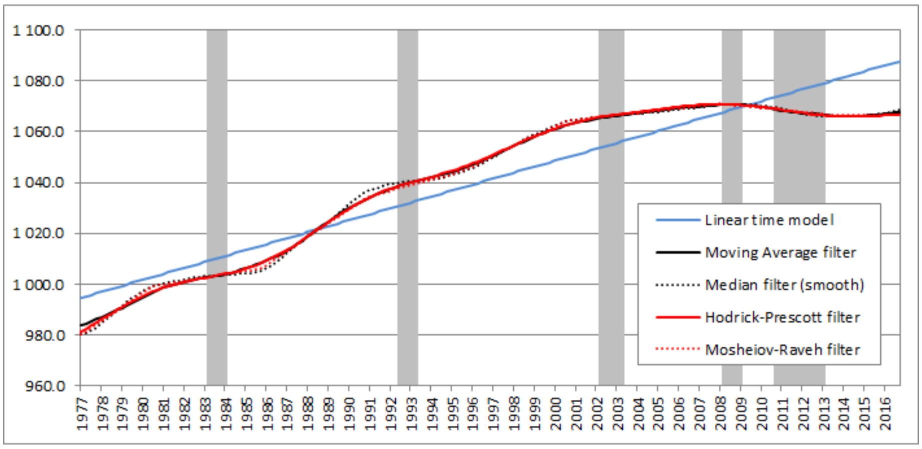
The MR filter also has this fitting property, in contrast with other statistical techniques that either force a constant GDP trend growth, or model this growth as a function of other covariates, or even allowing it to vary according to predefined periods. Of course, some of the limitations of the HP filter are well known, especially its bias near the end of the sample period, which can severely affect the estimates of trend growth and cyclical component. This bias can be serious and therefore most researchers treat with great care the estimates of GDP trend growth and cyclical position near the end, usually the present, of the available data.

The most interesting feature of the MR filter when compared with the HP filter is that it provides a more robust estimate of the trend growth, only changing this estimate if the signal coming from the data is sufficiently strong. Thus, the MR filter could delay the recognition of a change in regime, but it would be robust in signaling a new trend once sufficient evidence is present in the data.

3 Main findings

The five statistical methods presented above to separate the data into trend, cyclical and error components were applied to the quarterly real (chain linked) national accounts of GDP, private consumption, gross fixed capital formation (GFCF) and employment level provided by the Portuguese National Statistics Office (INE) since the first quarter of 1995 till the fourth quarter of 2016. This seasonally adjusted data was complemented with long-term series from Banco de Portugal (BdP) to produce a set of 160 observations since the first quarter of 1977. Moreover, a procedure described in the appendix was used to date the Portuguese recessions by tracking the peaks and troughs in the various series.

In figure 1, we show the trend component of GDP according to the five methods. The linear time model provides substantially different results from the other four. Notice how recessions tend to slow growth in all other four methods. Moreover, notice how the four methods, bar linear trend, show a declining trend GDP since 2008, which coincides roughly with the Great Recession 2007-2009.



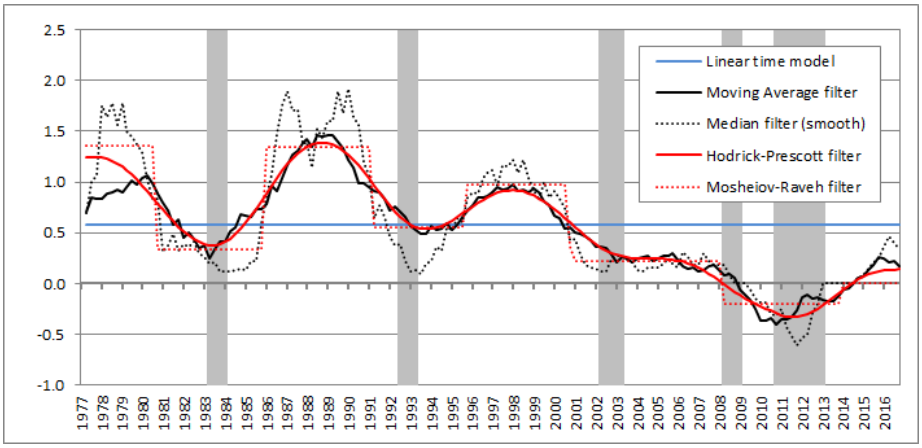
Notes: the Median trend was smoothed with a centered moving average of 5 quarters ($k = 2$) and the shaded regions denote the recession dates in table 4 (appendix).

Figure 1: Trend component of 100 times the log of GDP computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)

In figure 2, we plot the trend GDP growth, which brings more or less cyclical patterns accordingly with the model. What is most striking, however, is that after 2000 the declining estimate of GDP growth never returned to the highest values observed around the previous peaks. Moreover, trend GDP growth was smaller than the constant estimated by the linear time model from 2000 on. The private consumption, investment and employment follow similar patterns (see additional figures in annex). It is striking that the general shape of alternation between trend growth levels is quite similar across statistical methods, except the constant linear model.

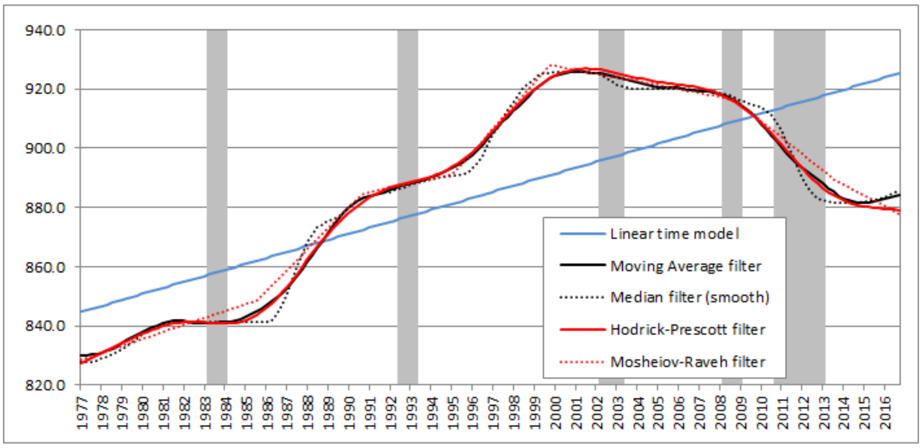
The behavior of investment is worth an additional note. According to most models, investment trend growth was negative between 2001 and 2014. Moreover, figure 3 shows how far from the levels observed in 2002 are the fixed investment nowadays. This is a good measure for the severity of slowdown in investment observed for close to 16 years (from 2002 to the present), a much extended period indeed.

An immediate approach is to interpret the different trend growth levels as different natural periods of the Portuguese economy. We do it in table 1, which proved to be a complement of the recessions dated in table 4 (appendix). While all filtering techniques (moving average, median filter, HP and MR) provide



Note: the shaded regions denote the recession dates in table 4 (appendix).

Figure 2: First difference of the trend component of 100 times the log of GDP computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



Notes: the Median trend was smoothed with a centered moving average of 5 quarters ($k = 2$) and the shaded regions denote the recession dates in table 4 (appendix).

Figure 3: Trend component of 100 times the log of GFCF computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)

similar estimates for GDP trend growth that highlight different periods, the MR filter provides this dating in a very natural fashion. Thus, we identify eight distinctive periods for the evolution of the Portuguese economy since 1977:

Period	**GDP**	**Consumption**	**GFCF**	**Employment**
1	1980Q3	1981Q3	-	1980Q2
2	1985Q4	1985Q4	1985Q3	1986Q2
3	1991Q1	1991Q4	1990Q4	1991Q1
4	1995Q3	1996Q3	1995Q1	1995Q4
5	2000Q3	2000Q4	1999Q4	2001Q2
6	2008Q1	2008Q1	2008Q4	2008Q2
7	2013Q4	2013Q4	2013Q3	2013Q4
8	tbd	tbd	tbd	tbd

tbd: to be determined.

Table 1: Portuguese economic periods as determined by the MR filter (end quarter)

The first period, ending in late 1980, represents a strong expansionary period (average growth of 1.4% per quarter). It was followed by a period of low growth (average growth of 0.3%) which ends in late 1985 and includes a short recession. The third period is again strongly expansionary (average quarterly growth of 1.4%) until 1991. The fourth period, which includes a small recession at the beginning, has a low growth but around the overall average of 0.6% quarter over quarter change, given by the linear time trend model.

Therefore, the MR filter appears to capture a broad concept of strong growth and low patch with a similar interpretation that is often used to characterize expansions and contractions. However, the remaining four periods show a very different alternation of trend growth. The apparent strong growth in period 5 (1.0% per quarter) is moderate by historical standards, but is still the strongest of the following three periods. The sixth period (average growth of 0.2%) is unusually long (seven years), but it was not followed by a period of strong growth. In effect, trend GDP growth becomes negative in period 7 (-0.2% per quarter) and includes two recessions. So it can be classified as a period with a double dip recession. This is the period with the weakest growth of the Portuguese economic history after 1974. The current period 8 started in late 2013 (average trend growth of 0.1%) which can be considered as an expansionary period, but much weaker that all others on record.

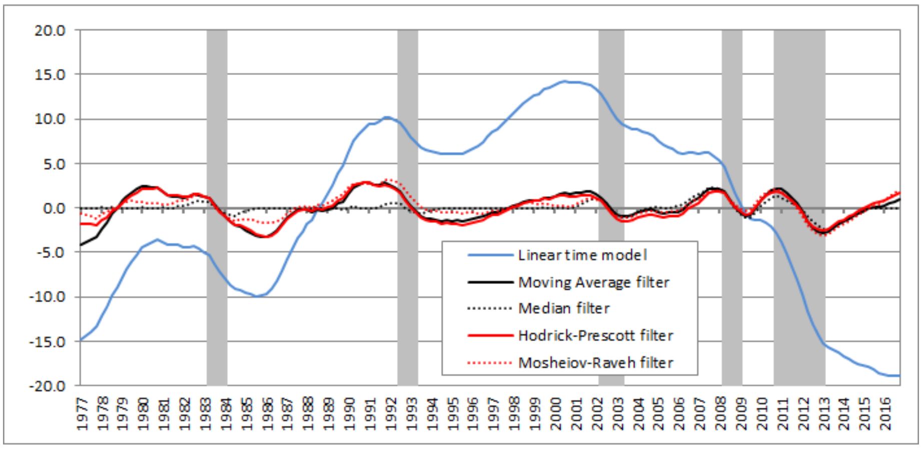
These eight periods are visible on series other than GDP. In some series, two periods might be considered as one single period (e.g. the first and second periods for investment, see table 1). Moreover, the exact beginning of the strong and the weak spots are not necessarily estimated to begin and end on the same quarter by all data series analyzed. However, they tell the same story from a qualitative point of view.

This description of the results shows the great power of the MR filter to identify natural periods of differential growth on the data series. These periods show up quite naturally, and the weak periods usually include one or two recessions. So, the MR filter can be used to that purpose. This type of classification is useful for policy since weak periods of growth should recommend fiscal stimulus and strong periods of growth should counsel additional debt repayments and savings.

One can argue that HP filter provides a similar story. However, the beginning and ending of the periods is less obvious (associated with inflexion points in figure 2), and the HP trend growth must be averaged to characterize each period directly signaled by the MR filter.

The five methods might provide very different estimates of the trend GDP growth. Using the last quarter of the data, the range of this growth is between 0.01% for the MR filter and 0.63% for the MED filter. Models using Box-Jenkins type procedures with the entire data set will tend to anchor medium term predictions around 0.6% change per quarter, plus eventually some short-term cyclical recovery. Nevertheless, the MR filter is quite conservative due to its robust estimation properties: changes of its estimated trend growth require stronger signals from data than pure noise or cyclical recovery.

Note that the linear model generates an extremely different pattern for the cyclical component, because it assumes a constant trend growth (see figure 4). The other filters generate fairly similar cycles except near the begin and end points of the sample.



Notes: the cycles were smoothed with a centered moving average of 5 quarters ($k = 2$) and the shaded regions denote the recession dates in table 4 (appendix).

Figure 4: Cyclical component of 100 times the log of GDP computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)

A remarkable feature of the nonlinear filtering techniques is that they produce cycle components less volatile than the difference between the original data and the HP trend, a suggested by the table 2. This result is particularly evident for the median filter and it is a direct consequence of the ability of nonlinear filters to capture sharp swifts in the trend growth. Thus, the cyclical components associated with nonlinear filters (including MR) are less contaminated with noise and changes other than pure cyclical effects. As already illustrated by the figure 4 for GDP, the cyclical components of the MED filter coincides almost exactly with the recessions dated in table 4 (appendix) with the Bry and Boschan algorithm [2, 14] which is a direct consequence of that characteristic of nonlinear filtering in general. Nevertheless, the MR filter realizes a cycle component that is more comparable with the HP cycle than MED cycle.

Component	**Linear**	**MA**	**MED**	**HP**	**MR**
GDP	9.84	1.78	0.89	1.65	1.44
Private consumption	11.87	2.21	1.08	2.04	1.94
Investment (GFCF)	23.08	5.74	4.21	5.61	6.67
Employment	4.84	1.18	0.80	1.15	0.90

Table 2: Standard deviation of the cyclical component of the Portuguese quarterly real GDP, private consumption, GFCF and employment computed with the linear time model, Moving Average (MA), Median (MED) Hodrick-Prescott (HP) and Mosheiov-Raveh (MR) filters

4 Conclusion

Accessing economic conditions overtime requires separating trend, cyclical and noise components from time series. This task is often model dependent. Simple detrending models assumes a secular trend that is fixed over time, thus all shocks are described as cyclical or temporary within that kind of framework. However, technology and other fundamentals will likely change trend growth overtime. So, the trend of an economy can be better described as a slow changing movement over time rather than a fixed drift. The Hodrick-Prescott [13] filter is very popular in part because it can estimate that kind of non-constant but smooth trend growth, producing insightful business cycles.

However, the HP and other linear filtering techniques generate trends with regular and predictable patterns that pass everything else to the stationary component, including high-frequencies and shocks other than pure cyclical effects. In practice, they produce 'artificial' trend-cycle decompositions that might not reflect the data-generating processes.

Robust nonlinear filters could perform better than linear filters in capturing discrete shifts in the trend growth of economic series. For example, the median filter provides noise-attenuation and it has been effective in signaling recessions. Nevertheless, its trend is noisy even when averaged across adjacent time periods. We found that the method proposed by Mosheiov and Raveh [18] keeps the robust statistical proprieties of the median filter, while generating stable trend growth estimates lend themselves to insightful interpretations of the different time periods of an economy.

The distinctive feature of the MR filter is the piecewise nature of its trend growth. This unique characteristic might be useful to estimate the trend growth of the economy and to highlight natural periods of differential growth on the data series. In fact, the MR filter is conceptually similar to HP filter but uses robust statistics anchored on the median and absolutes deviations of the data. Thus, the MR filter provides a more robust estimate of the trend growth, only changing this estimate if the signal coming from the data is sufficiently strong. Additionally, it provides potentially better measure of the cyclical position and as such a measure of the slack or overheating of the economy.

With the help of the MR filter, we found eight distinctive periods for the evolution of the Portuguese economy since 1977. These periods show up quite naturally, and the weak periods might include one or two recessions for some variables such investment or employment. In particular, the sixth period (2000Q4-2008Q1) was unusually long, has a moderate average trend GDP growth of only 0.2% per quarter, and it was not followed by a period of strong growth yet.

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Appendix: Dating the Portuguese recessions

A *recession* may be defined as a significant decline in the level of economic activity, not confined to one sector but spread across the economy, usually visible in two or more consecutive quarters of negative growth of GDP, employment and other measures of aggregate economic activity [3]. Thus, a recession starts right after the economy reaches a *peak* and deems to end when growth resumes in GDP and other key measures of economic activity, that is, after a *trough*, when starts the *recovery* till the next peak.

In most applications, peaks and troughs were tracked using the well-known Bry and Boschan [2] dating algorithm that roughly identified the same business cycle reference dates of the National Bureau of Economic Research (NBER) for the United States of America [8]. In this framework, a *peak* occurred in quarter t for a previously smoothed variable y if

$$y_{t-2}, y_{t-1} \leq y_t \geq y_{t+1}, y_{t+2}. \tag{20}$$

Similarly, a *trough* occurred in quarter t if

$$y_{t-2}, y_{t-1} \geq y_t \leq y_{t+1}, y_{t+2}. \tag{21}$$

For quarterly data, the minimum peak-to-trough (trough-to-peak) period is two quarters and peak-to-peak (trough-to-trough) is six quarters.

Here, we applied the `MatLab` implementation of the Bry and Boschan algorithm developed by Robert Inklaar [14] to the macroeconomic variables that the Euro Area Business Cycle Dating Committee from the Centre for Economic Policy Research (CEPR) had used to fix those turning points for the euro-area: GDP (chain linked volumes), household and other private final consumption expenditure (chain linked volumes), investment (gross fixed capital formation - GFCF, chain linked volumes) and employment (number of persons) [3].

Turning points	**GDP**	**Consumption**	**GFCF**	**Employment**
Peak (P)	1983Q1	1982Q2	1982Q1	1982Q2
Trough (T)	1984Q1	1985Q1	1985Q2	1983Q2
Peak (P)	1992Q2	-	1992Q1	1992Q1
Trough (T)	1993Q2	-	1993Q4	1993Q3
Peak (P)	2002Q1	2002Q1	2001Q4	2002Q2
Trough (T)	2003Q2	2003Q2	2003Q4	2005Q3
Peak (P)	2008Q1	2008Q1	2008Q1	2008Q2
Trough (T)	2009Q1	2009Q2	-	-
Peak (P)	2010Q3	2010Q4	-	-
Trough (T)	2012Q4	2013Q1	2013Q1	2013Q1

Table 3: Business cycle reference dates (peaks and troughs) based on quarterly real GDP, private consumption, investment (GFCF) and employment in Portugal since 1977

A total of five peaks was dated for the Portuguese GDP since 1977Q1: 1983Q1, 1992Q2, 2002Q1, 2008Q1 and 2010Q3 (see table 3). In two of theses cases (2002Q1 and 2008Q1), peaks were observed also in private consumption expenditure. Typically, investment peaks occurred before GDP local maxima such as in 1982Q1, 1992Q1 and 2001Q4, suggesting that changes in GFCF could signal the turning points of the product with a lag of at least one quarter. The employment followed a less predictable pattern in the sense that peaks can happen either before (1982Q2, 1992Q1) or after (2002Q2, 2008Q2) GDP high-levels.

Dating troughs is a tricky task because low-levels were more sparse within the variables of interest. For example, the 1993Q2 GDP trough was observed with one and two quarters of delay in employment and GFCF, respectively, and it was not tracked by the algorithm [14] for private consumption. Thus, the process of dating a trough might be delayed in order to assure a broadly consistent end of recession. In most cases, the date of the GDP trough sounds well to fix that end, but this might not be true for 2012Q4 (see table 4). In this particular, severe recession, the trough was fixed with one quarter of delay (2013Q1) from GDP local minimum in order to accommodate the low-levels observed in the other variables subsequently. The pertinence of this delaying was confirmed by a dating exercise with the monthly coincident indicator for the Portuguese economic activity [20].

Recession	**Peak**	**Trough**	**Duration**	**Depth (*)**	**Output loss**
I	1983Q1	1984Q1	4	-4.1%	-2.3%
II	1992Q2	1993Q2	4	-2.2%	-1.5%
III	2002Q1	2003Q2	5	-2.2%	-2.4%
IV	2008Q1	2009Q1	4	-1.8%	-4.4%
V	2010Q3	2013Q1	10	-3.5%	-8.4%

(*) Minimum output gap during or after each recession, computed with the HP filter.

Table 4: Recessions in Portugal since 1977

Another interesting feature of the dating exercise described in the previous tables is that troughs are not observed around 2009Q1 for investment and employment. Thus, the recessions IV and V might be considered as an unique contraction episode that lasted between 2008Q2 and 2013Q1.

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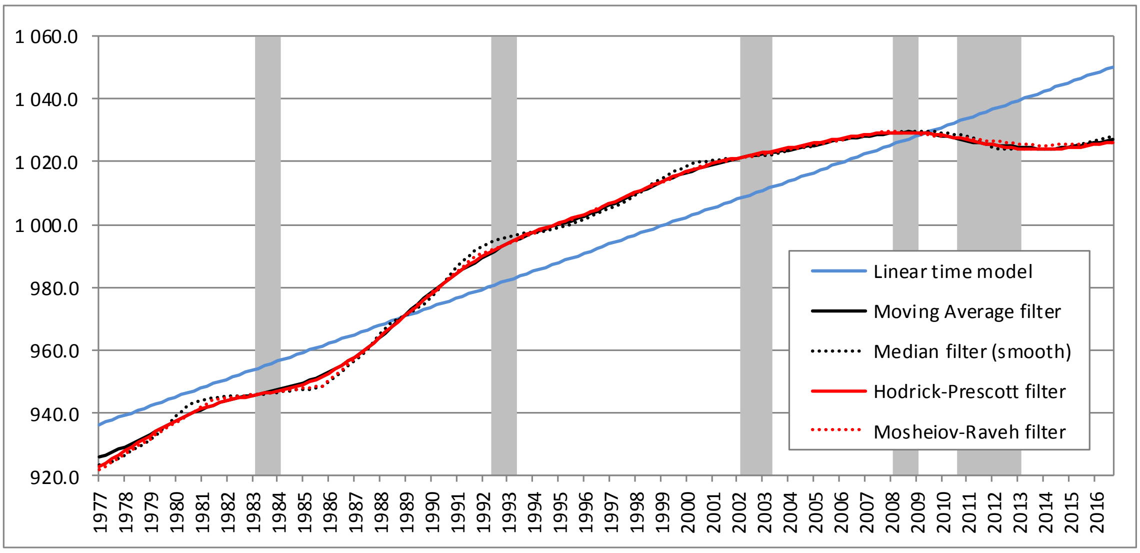
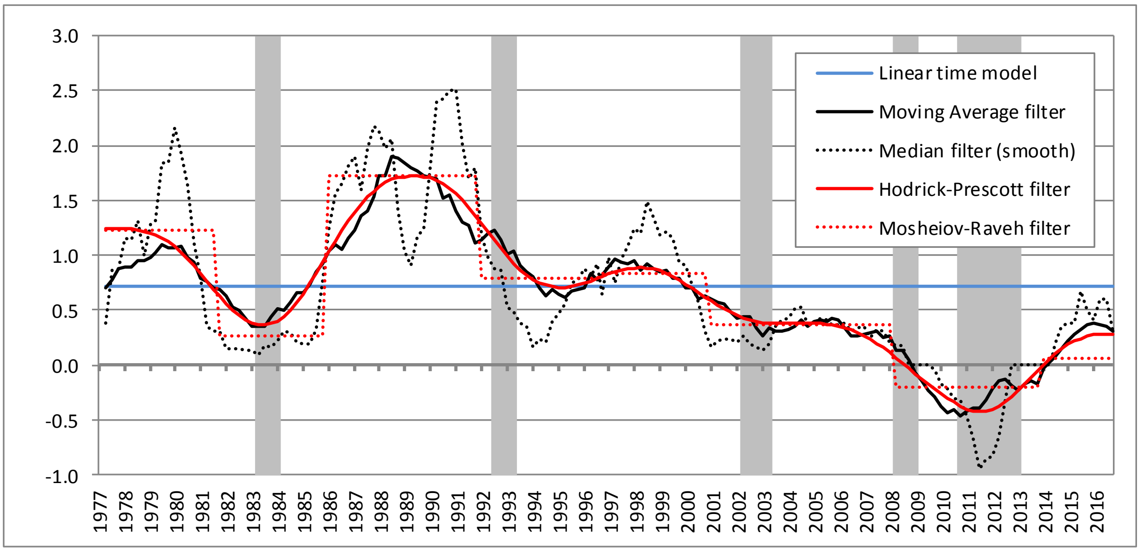
Annex: additional figures

Figure A.1 – Trend component of 100 times the log of private consumption computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



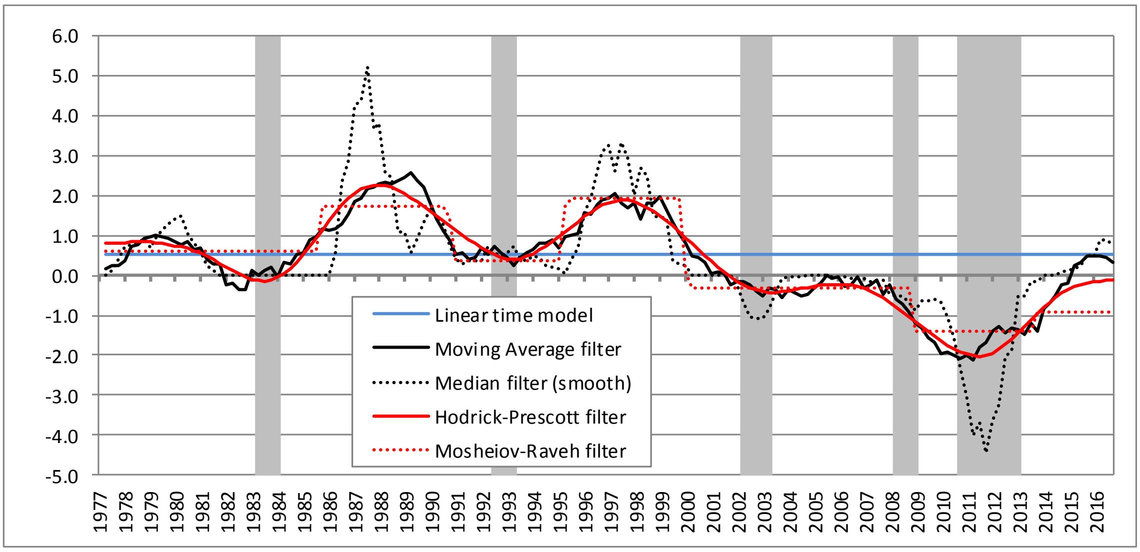
Notes: the Median trend was smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.2 – First difference of the trend component of 100 times the log of private consumption computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



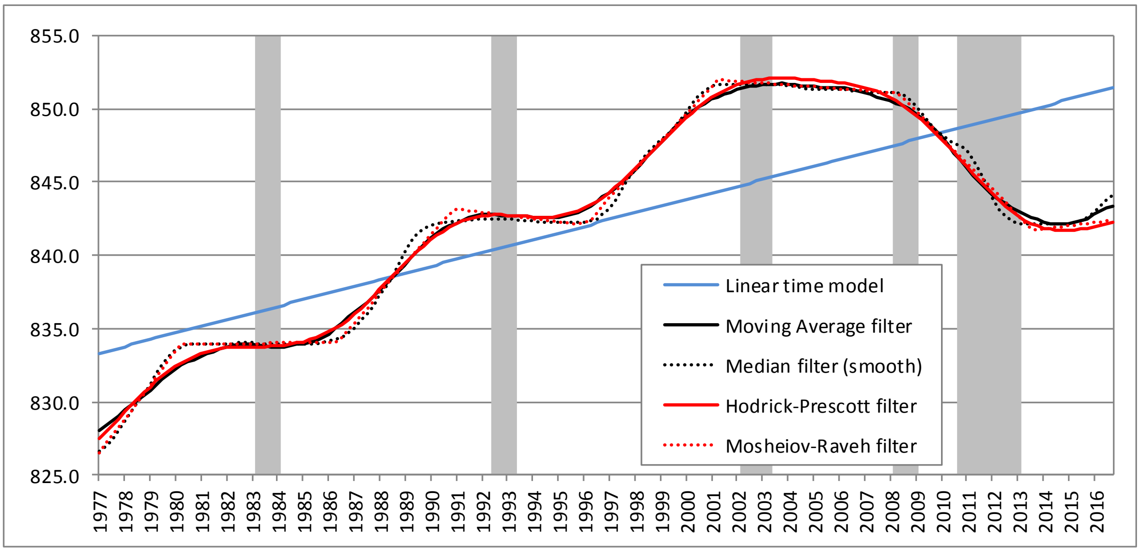
Note: the shaded regions denote the recession dates in table 4.

Figure A.3 – First difference of the trend component of 100 times the log of GFCF computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



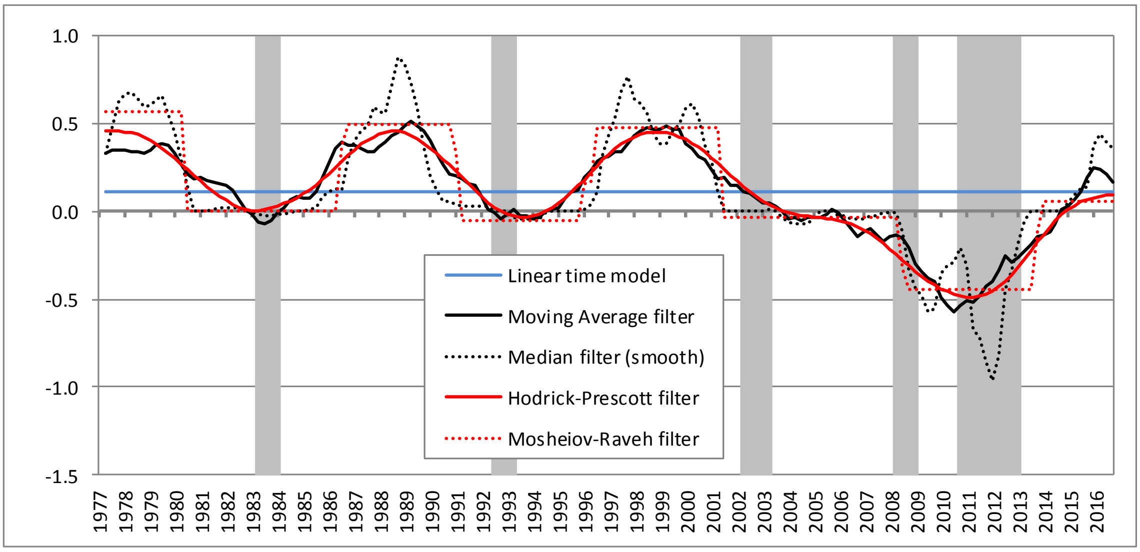
Note: the shaded regions denote the recession dates in table 4.

Figure A.4 – Trend component of 100 times the log of employment computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



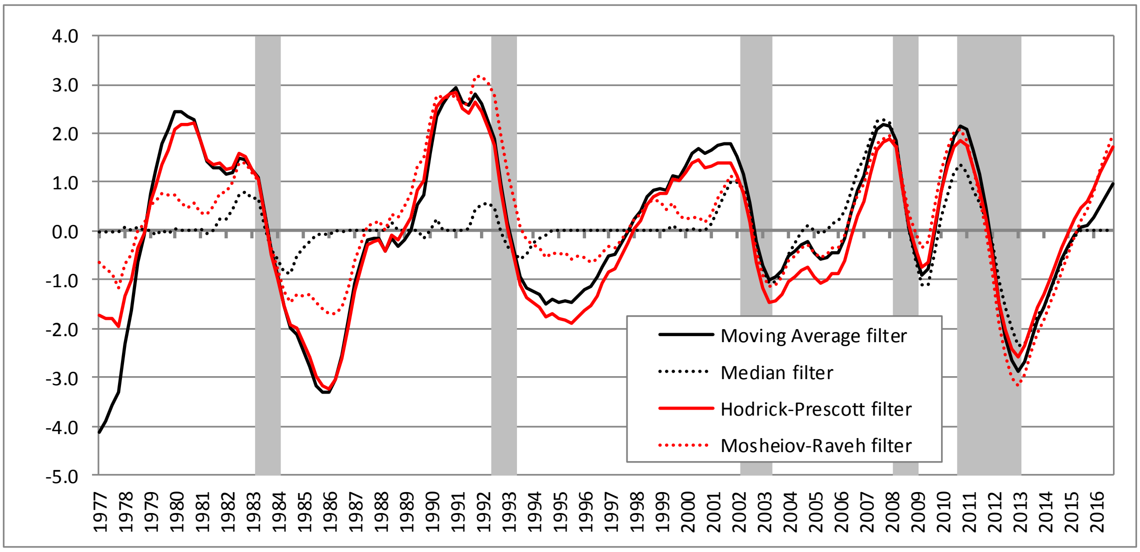
Notes: the Median trend was smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.5 – First difference of the trend component of 100 times the log of employment computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



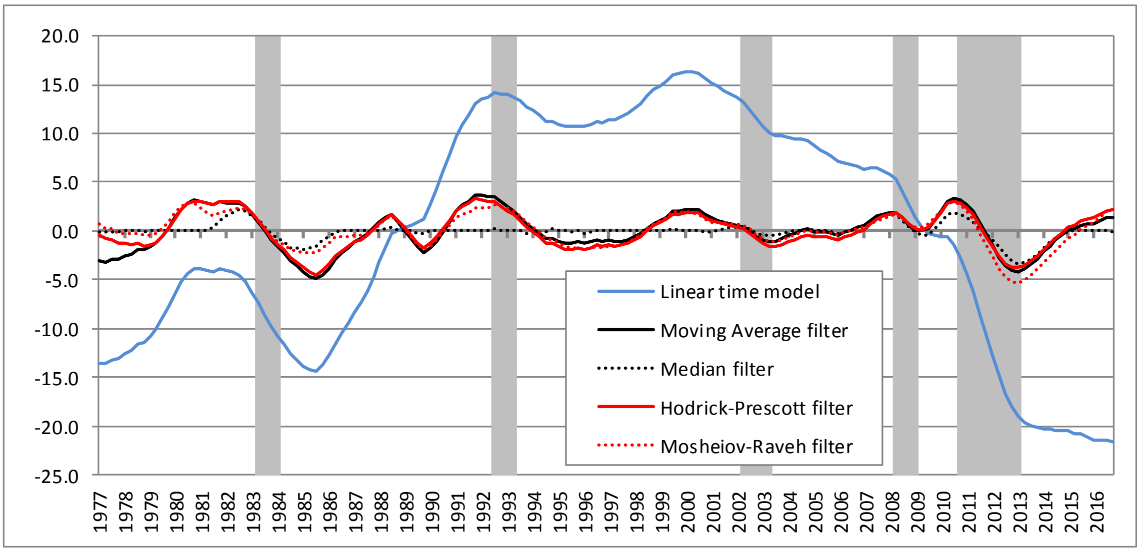
Note: the shaded regions denote the recession dates in table 4.

Figure A.6 – Cyclical component of 100 times the log of GDP computed with the Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



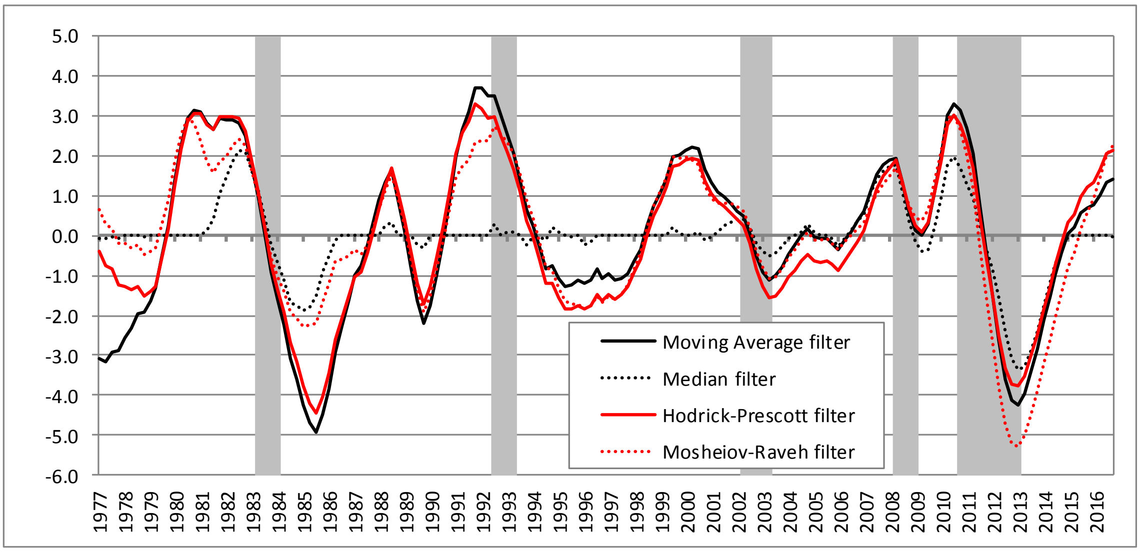
Notes: the cycles were smoothed with a centered moving average of 5 quarters ($k = 2$) and the shaded regions denote the recession dates in table 4.

Figure A.7 – Cyclical component of 100 times the log of private consumption computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



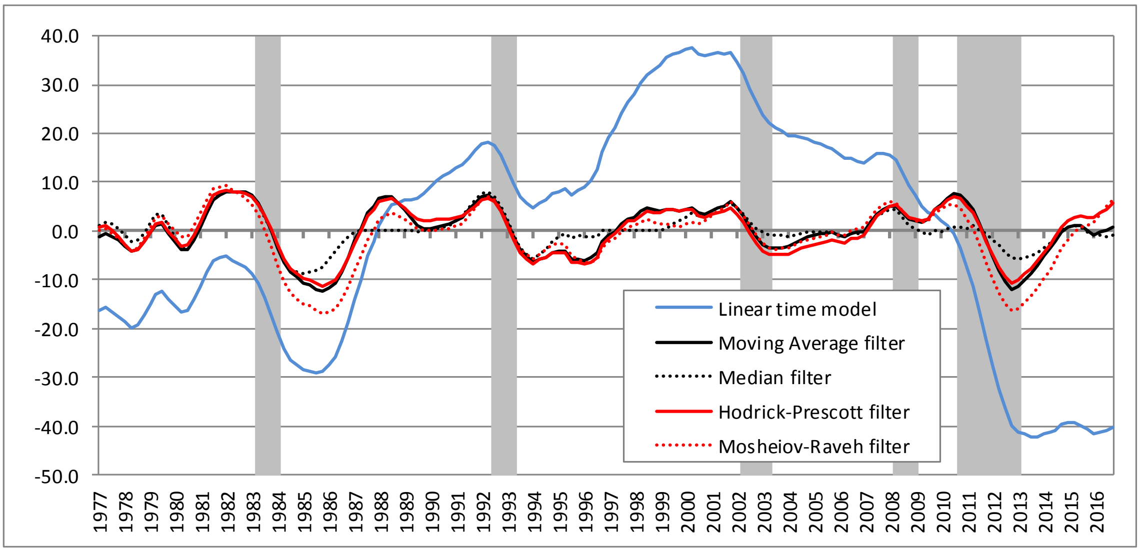
Notes: the cycles were smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.8 – Cyclical component of 100 times the log of private consumption computed with the Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



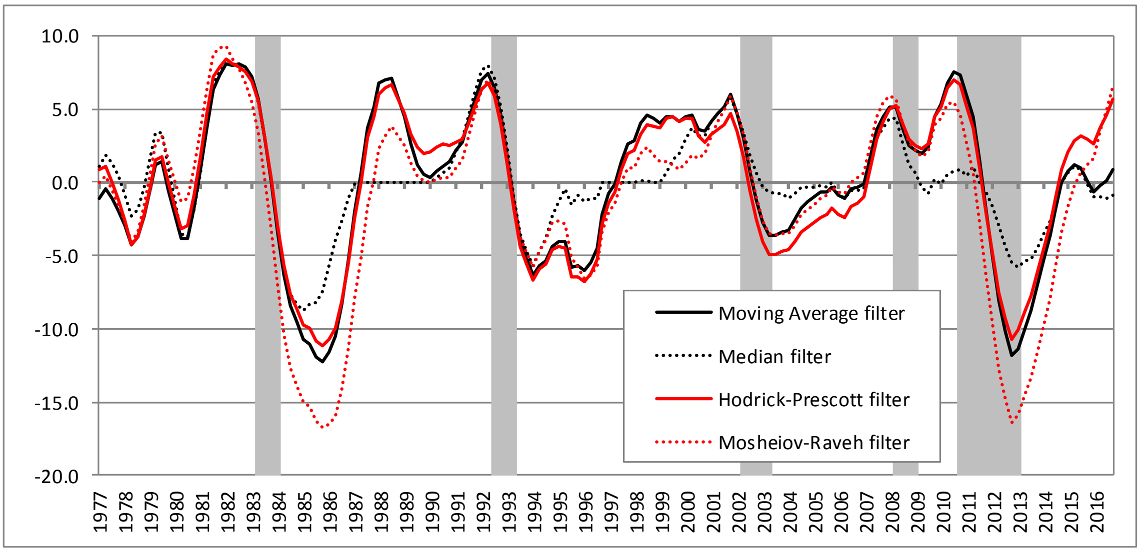
Notes: the cycles were smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.9 – Cyclical component of 100 times the log of GFCF computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



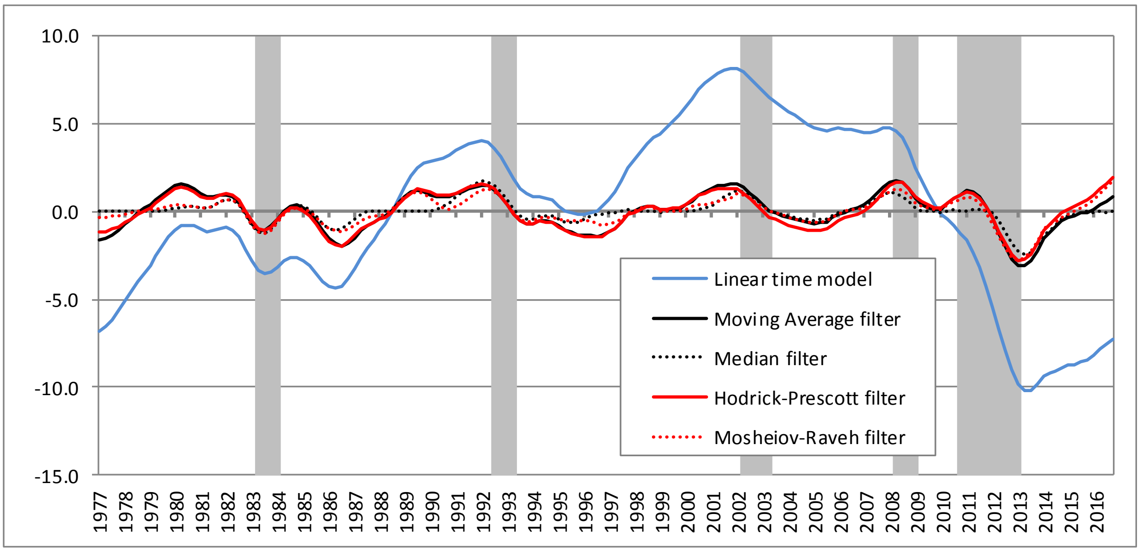
Notes: the cycles were smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.10 – Cyclical component of 100 times the log of GFCF computed with the Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



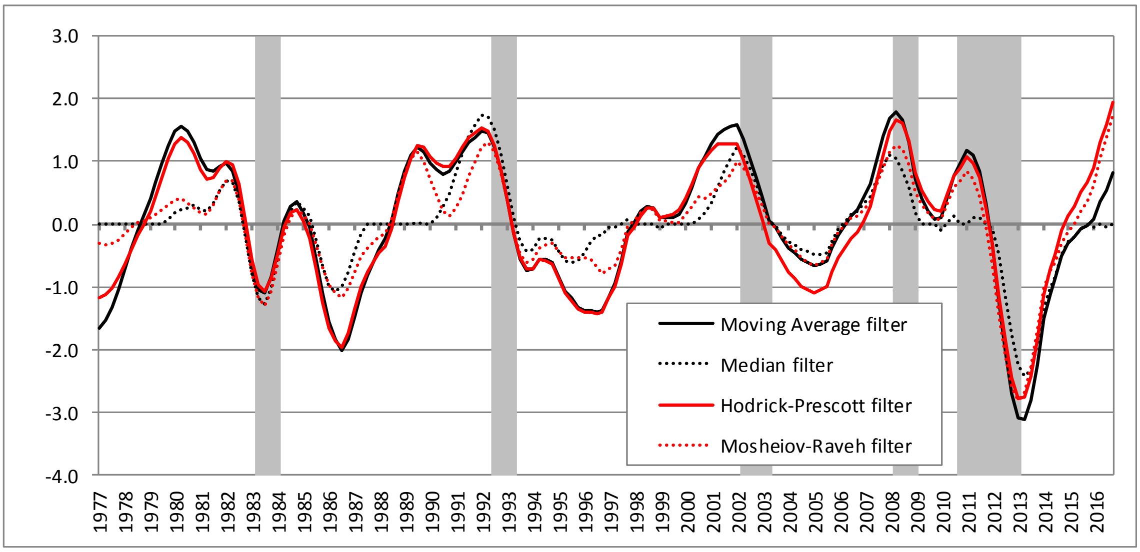
Notes: the cycles were smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.11 – Cyclical component of 100 times the log of employment computed with the linear time model (blue), Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



Notes: the cycles were smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.

Figure A.12 – Cyclical component of 100 times the log of employment computed with the Moving Average (black), Median (black dashed), Hodrick-Prescott (red) and Mosheiov-Raveh (red dashed) filters (Portugal, 1977Q1-2016Q4)



Notes: the cycles were smoothed with a centered moving average of 5 quarters (k = 2) and the shaded regions denote the recession dates in table 4.